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A Computational Method for Self-Adjoint Singular Perturbation Problems Using Quintic Spline

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Abstract—Singularly perturbed self-adjoint boundary-value problems are considered in this article. A difference scheme based on quintic spline is proposed. This scheme is applied to the subproblems obtained from the given problem by dividing the whole domain into nonoverlapping subdomains. The proposed scheme is of fourth-order convergent and more suitable for parallel computers. Stability and convergence of the method are discussed. Numerical examples are provided to show the efficiency and accuracy.

Keywords—Singular perturbation problem, Boundary layers, Asymptotic approximation, Domain decomposition, Difference schemes, Quintic spline.

1. INTRODUCTION

Singular perturbation problems (SPPs) arise in several branches of engineering and applied mathematics which include fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, etc. The presence of small parameter(s) in these problems prevents us from obtaining satisfactory numerical solutions. It is a well-known fact that the solutions of SPPs have a multiscale character. That is, there are thin layer(s) where the solution varies very rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. Various finite-difference schemes have been proposed in literature to guarantee stability of the schemes for all values of the perturbation parameter. Careful examination of numerical results from such schemes on uniform grids shows that, for fixed (small) values of the perturbation parameter, the maximum pointwise error usually increases as the mesh is refined, because of the presence of the boundary or interior layer, until the mesh diameter is comparable in size to the parameter. This behavior is clearly unsatisfactory. Therefore, a separate treatment is necessary to deal with such problems.

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In this article, we consider the following singularly perturbed self-adjoint boundary-value problem (BVP),

$$Lu(x) \equiv -\varepsilon^2 u''(x) + b(x)u(x) = f(x), \qquad x \in D = (0,1), \tag{1.1}$$

$$u(0) = A, \qquad u(1) = B, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, b and f are sufficiently smooth functions, such that $b(x) \geq \beta > 0$ on $\bar{D} = [0,1]$. Under these assumptions, the BVP (1.1),(1.2) possesses a unique solution $u(x) \in \mathcal{C}^2(\bar{D})$. In general, the solution $u(x)$ may exhibit two boundary layers of exponential type at both end points $x = 0, 1$.

To solve these types of problems various methods are proposed in the literature, more details can be found in [1,2]. Natesan *et al.* [3,4] proposed a domain decomposition method for singularly perturbed BVPs, and applied the exponentially-fitted difference (EFD) scheme in the boundary layer region, and classical finite-difference scheme in the regular region. Recently, the authors implemented a similar method with more number of subdomains in a parallel processing machine [5]. Kadalbajoo and Bawa [6,7] used cubic spline on variable mesh for solving SPPs. In [8], the authors provided a booster method, which incorporates an asymptotic expansion into any numerical method and give higher-order accuracy. Jayakumar [9] treated numerically the above problem by dividing the domain into nonoverlapping subdomains and used EFD and classical finite-difference scheme to solve the problems.

The main contribution of the present paper is to use higher-order spline schemes of regular problems to singular perturbation problems of the form (1.1),(1.2) after suitable modifications. More precisely, we divide the domain into three subdomains: two boundary layer subdomains, and one regular subdomain, and we convert the boundary layer problems to a regular one by proper transformations using stretching variables, and then we apply a difference scheme based on quintic splines to obtain the numerical solution in the three subdomains. To obtain the boundary condition at the transition points, we use an asymptotic approximate solution. By this way, we obtain higher-order accuracy $O(h^4)$, as well as utilize the parallel computers to reduce the computation time, because the boundary layers and regular subdomain problems are independent of each other.

We organize the article in the following style. Difference schemes based on quintic splines are proposed in Section 2. The present method is provided in Section 3. Section 4 presents error estimates and convergence analysis. Numerical experiments are given in Section 5 and the paper concludes with a discussion.

2. DIFFERENCE SCHEME BASED ON QUINTIC SPLINE

In this section, we derive a difference scheme based on quintic splines. For this, we consider the following two-point BVP without any small parameter,

$$u''(x) = q(x)\,u(x) + r(x), \qquad x \in (a,b), \tag{2.1}$$

$$u(a) = \gamma_1, \qquad u(b) = \gamma_2, \tag{2.2}$$

where the coefficients q, r are sufficiently smooth functions, and γ_1, γ_2 are given constants.

To derive the difference scheme, let $S(x)$ be a quintic spline defined on the interval $[a,b]$ with equally spaced knots,

$$x_j = a + jh, \qquad j = 0, \dots, N, \tag{2.3}$$

where $h = (b-a)/N$.

Then, $S(x)$ satisfies the following.

(i) $S(x)$ is a polynomial of degree at most five in each subinterval $[x_{j-1}, x_j]$.

(ii) The first four derivatives $S^{(1)}(x)$, $S^{(2)}(x)$, $S^{(3)}(x)$, and $S^{(4)}(x)$ are continuous in $[a,b]$.

Let u_j be an approximation to $u(x_j)$ obtained by the quintic spline $S(x_j)$. Moreover, for $j = 0, \ldots, N+1$, we use the following notations,

$$\begin{gathered} S(x_j) = u_j, \qquad S^{(1)}(x_j) = m_j, \qquad S^{(2)}(x_j) = M_j, \\ S^{(3)}(x_j) = n_j, \qquad S^{(4)}(x_j) = N_j. \end{gathered} \tag{2.4}$$

It may be noted that the $S_j(x)$, $j = 1, \ldots, N$ can be defined on the interval $[x_{j-1}, x_j]$ by integrating the following equation,

$$S_j^{(4)}(x) = \frac{1}{h}\left[N_{j-1}(x_j - x) + N_j(x - x_{j-1})\right] \tag{2.5}$$

four times with respect to x. Precisely,

$$S_j(x) = \frac{1}{120h}\left[N_{j-1}(x_j - x)^5 + N_j(x - x_{j-1})^5\right] + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D,$$

where A, B, C, and D are the constants of integration. To calculate these constants of integrations, we use the following conditions,

$$S_j(x_j) = u_j, \qquad S_j^{(2)}(x_j) = M_j, \qquad S_j(x_{j-1}) = u_{j-1}, \qquad S_j^{(2)}(x_{j-1}) = M_{j-1}. \tag{2.6}$$

The identities involving functions and its derivatives of quintic splines for the solution of (2.1),(2.2) can be written as, for $j = 2, \ldots, N-2$,

$$\begin{aligned} m_{j-2} + 26m_{j-1} + 66m_j + 26m_{j+1} + m_{j+2} &= \frac{5}{h}\left[-u_{j-2} - 10u_{j-1} + 10u_{j+1} + u_{j+2}\right], \\ M_{j-2} + 26M_{j-1} + 66M_j + 26M_{j+1} + M_{j+2} &= \frac{20}{h^2}\left[u_{j-2} + 2u_{j-1} - 6u_j + 2u_{j+1} + u_{j+2}\right], \\ n_{j-2} + 26n_{j-1} + 66n_j + 26n_{j+1} + n_{j+2} &= \frac{60}{h^3}\left[u_{j-2} + 2u_{j-1} - 2u_{j+1} + u_{j+2}\right], \\ N_{j-2} + 26N_{j-1} + 66N_j + 26N_{j+1} + N_{j+2} &= \frac{120}{h^4}\left[u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}\right]. \end{aligned} \tag{2.7}$$

Relations (2.7) can be derived from Ahlberg and Walsh [10]. For example, we take the following one-sided first-order derivatives,

$$S^{(1)}(x_-) = \frac{1}{h}(u_j - u_{j-1}) + \frac{h}{6}(2M_j + M_{j-1}) - \frac{h^3}{360}(8N_j + 7N_{j-1}), \qquad j = 1, \ldots, N, \tag{2.8}$$

and

$$S^{(1)}(x_+) = \frac{1}{h}(u_{j+1} - u_j) - \frac{h}{6}(M_j + M_{j+1}) - \frac{h^3}{360}(8N_j + 7N_{j+1}), \qquad j = 0, \ldots, N-1. \tag{2.9}$$

Now, the continuity of the first derivative implies that

$$\begin{gathered} (M_{j-1} + 4M_j + M_{j+1}) = \frac{6}{h^2}(u_{j-1} - 2u_j + u_{j+1}) \\ + \frac{h^2}{60}(7N_{j-1} + 16N_j + 7N_{j+1}), \qquad j = 1, \ldots, N-1. \end{gathered} \tag{2.10}$$

Similarly, from the continuity of third derivative, we have

$$(M_{j-1} - 2M_j + M_{j+1}) = \frac{h^2}{6}(N_{j-1} + 4N_j + N_{j+1}), \qquad j = 1, \ldots, N-1. \tag{2.11}$$

Subtracting (2.10) from (2.11) and dividing by six, we obtain

$$M_j = \frac{1}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) - \frac{h^2}{120}\left(N_{j-1} + 8N_j + N_{j+1}\right), \qquad j = 1, \dots, N-1. \tag{2.12}$$

Elimination of M_j from (2.11) and (2.12) leads to fourth relation of (2.7).

We consider the system given in the second equation of (2.7), as it involves the function $S_j(x_j)$ and its second derivative terms,

$$\begin{aligned} & u_{j-2} + 2u_{j-1} - 6u_j + 2u_{j+1} + u_{j+2} \\ &= \frac{h^2}{20}\left(M_{j-2} + 26M_{j-1} + 66M_j + 26M_{j+1} + M_{j+2}\right), \qquad j = 2, \dots, N-2. \end{aligned} \tag{2.13}$$

From (2.1) and (2.4), we can have

$$\begin{aligned} M_j &= q_j u_j + r_j, \qquad j = 1, \dots, N-1, \\ M_0 &= q_0 \gamma_1 + r_0, \qquad M_N = q_N \gamma_2 + r_N. \end{aligned} \tag{2.14}$$

Substituting the values of M_j from (2.14) into (2.13), we obtain

$$\begin{aligned} u_{j-2} + 2u_{j-1} - 6u_j + 2u_{j+1} + u_{j+2} &= \frac{h^2}{20}\left[\left(q_{j-2}u_{j-2} + r_{j-2}\right) + 26\left(q_{j-1}u_{j-1} + r_{j-1}\right)\right. \\ &\left. +66\left(q_j u_j + r_j\right) + 26\left(q_{j+1}u_{j+1} + r_{j+1}\right) + \left(q_{j+2}u_{j+2} + r_{j+2}\right)\right], \qquad j = 2, \dots, N-2. \end{aligned} \tag{2.15}$$

We have $(N-3)$ equations in $(N-1)$ unknowns u_j, and we need four additional relations. Since the boundary conditions give relations determining $u_0 = \gamma_1$, and $u_N = \gamma_2$, we need only two more equations. This is achieved using quartic splines in the neighborhood of the two end points. The relation obtained near the left boundary point $x_0 = a$ is

$$\begin{aligned} 4u_0 - 7u_1 + 2u_2 + u_3 &= \frac{h^2}{12}\left[4\left(q_0u_0 + r_0\right) + 41\left(q_1u_1 + r_1\right)\right. \\ &\left. +14\left(q_2u_2 + r_2\right) + \left(q_3u_3 + r_3\right)\right]. \end{aligned} \tag{2.16}$$

Similarly, the relation obtained near the right boundary point $x_N = b$ is

$$\begin{aligned} u_{N-3} + 2u_{N-2} - 7u_{N-1} + 4u_N &= \frac{h^2}{12}\left[\left(q_{N-3}u_{N-3} + r_{N-3}\right) + 14\left(q_{N-2}u_{N-2} + r_{N-2}\right)\right. \\ &\left. +41\left(q_{N-1}u_{N-1} + r_{N-1}\right) + 4\left(q_Nu_N + r_N\right)\right]. \end{aligned} \tag{2.17}$$

Rearranging the relations (2.15)–(2.17), we obtain the following difference scheme,

$$\begin{aligned} & -\left[7 + \frac{41h^2}{12}q_1\right]u_1 + \left[2 - \frac{14h^2}{12}q_2\right]u_2 + \left[1 - \frac{h^2}{12}q_3\right]u_3 \\ &\quad = -4\left[1 - \frac{h^2}{12}q_0\right]\gamma_1 + \frac{h^2}{12}\left[4r_0 + 41r_1 + 14r_2 + r_3\right], \\[2ex] & \left[1 - \frac{h^2}{20}q_{i-2}\right]u_{i-2} + \left[2 - \frac{26h^2}{20}q_{i-1}\right]u_{i-1} - \left[6 + \frac{66h^2}{20}q_i\right]u_i \\ &\quad + \left[2 - \frac{26h^2}{20}q_{i+1}\right]u_{i+1} + \left[1 - \frac{h^2}{20}q_{i+2}\right]u_{i+2} \\ &\quad = \frac{h^2}{20}\left[r_{i-2} + 26r_{i-1} + 66r_i + 26r_{i+1} + r_{i+2}\right], \qquad i = 2, \dots, N-2, \\[2ex] & \left[1 - \frac{h^2}{12}q_{N-3}\right]u_{N-3} + \left[2 - \frac{14h^2}{12}q_{N-2}\right]u_{N-2} - \left[7 + \frac{41h^2}{12}q_{N-1}\right]u_{N-1} \\ &\quad = -4\left[1 - \frac{h^2}{12}q_N\right]\gamma_2 + \frac{h^2}{12}\left[r_{N-3} + 14r_{N-2} + 41r_{N-1} + 4r_N\right]. \end{aligned} \tag{2.18}$$

This system can be written in the following matrix form,

$$\left(J-\frac{h^2}{60}BQ\right)U=C+\frac{h^2}{60}BR, \tag{2.19}$$

where

$$J=\begin{bmatrix} -7 & 2 & 1 & & & & \\ 2 & -6 & 2 & 1 & & & \\ 1 & 2 & -6 & 2 & 1 & & \\ & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & 1 & 2 & -6 & 2 & 1 \\ & & & 1 & 2 & -6 & 2 \\ & & & & 1 & 2 & 7 \end{bmatrix}, \quad B=\begin{bmatrix} 205 & 70 & 5 & & & & \\ 78 & 198 & 78 & 3 & & & \\ 3 & 78 & 198 & 78 & 3 & & \\ & \cdot & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 3 & 78 & 198 & 78 & 3 \\ & & & 3 & 78 & 198 & 78 \\ & & & & 5 & 70 & 205 \end{bmatrix},$$

$$C=\begin{bmatrix} -4\gamma_1+\dfrac{h^2(q_0\gamma_1+r_0)}{3} \\ -\gamma_1+\dfrac{h^2(q_0\gamma_1+r_0)}{20} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -\gamma_2+\dfrac{h^2(q_N\gamma_2+r_N)}{20} \\ -4\gamma_2+\dfrac{h^2(q_N\gamma_2+r_N)}{3} \end{bmatrix}, \quad Q=\begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_{N-2} \\ q_{N-1} \end{bmatrix}, \quad \text{and} \quad R=\begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_{N-2} \\ r_{N-1} \end{bmatrix}.$$

3. DECOMPOSITION OF THE DOMAIN

We decompose the computational domain $\bar{D}=[0,1]$ into three subdomains, and then solve the differential equation (1.1), subject to different boundary conditions in each subdomain. Let $k=\ln(N)>0$, and $k\varepsilon$ be the width of the boundary layer(s) which is near at $x=0$, and $x=1$. More precisely, we divide the domain $\bar{D}$ into three nonoverlapping subdomains, as $D_1=[0,k\varepsilon]$, $D_2=[k\varepsilon,1-k\varepsilon]$, and $D_3=[1-k\varepsilon,1]$, such that $\bar{D}=D_1\cup D_2\cup D_3$. The subdomains D_1 and D_3 are called the boundary layer regions, whereas D_2 is known as the regular region.

The BVPs correspond to the left and right boundary layers are

$$-\varepsilon^2u''(x)+b(x)\,u(x)=f(x), \qquad x\in D_1=(0,k\varepsilon), \tag{3.1}$$

$$u(0)=A, \qquad u(k\varepsilon)=\tilde{A}, \tag{3.2}$$

and

$$-\varepsilon^2u''(x)+b(x)\,u(x)=f(x), \qquad x\in D_2=(1-k\varepsilon,1), \tag{3.3}$$

$$u(1-k\varepsilon)=\bar{B}, \qquad u(1)=B. \tag{3.4}$$

It is well known that the width of the boundary layers is $O(\varepsilon)$, in order to magnify the boundary layers, we use the stretching variable for the left and right boundary layers, respectively, by $\tau=x/\varepsilon$, and $\eta=(1-x)/\varepsilon$, and the transformed BVPs are given as

$$-U_1''(\tau)+B(\tau)\,U_1(\tau)=F(\tau), \qquad \tau\in(0,k), \tag{3.5}$$

$$U_1(0)=A, \qquad U_1(k)=\tilde{A}, \tag{3.6}$$

and

$$-U_2''(\eta) + B(\eta)\, U_2(\eta) = F(\eta), \qquad \eta \in (0,k), \tag{3.7}$$
$$U_2(0) = B, \qquad U_2(k) = \bar{B}, \tag{3.8}$$

where $U_1(\tau) = u(\tau\varepsilon)$, $U_2(\eta) = u(1+\eta\varepsilon)$, $B(\tau) = b(\tau\varepsilon)$, $B(\eta) = b(1+\eta\varepsilon)$, $F(\tau) = f(\tau\varepsilon)$, $F(\eta) = f(1+\eta\varepsilon)$.

The regular region BVP is given by

$$-\varepsilon^2 u''(x) + b(x)\, u(x) = f(x), \qquad x \in D_2 = (k\varepsilon, 1-k\varepsilon), \tag{3.9}$$
$$u(k\varepsilon) = \bar{A}, \qquad u(1-k\varepsilon) = \bar{B}. \tag{3.10}$$

To determine the boundary conditions at the transition points, we take the zero$^{\text{th}}$-order asymptotic approximation of the BVP (1.1),(1.2) given by

$$\tilde{u}(x) = u_0(x) + v_0(\tau) + w_0(\eta),$$

where $u_0(x) = f(x)/b(x)$ is the reduced problem solution, and v_0, and w_0 are, respectively, the left and right boundary layer correction terms,

$$v_0(\tau) = [A - u_0(0)] \exp\left(\frac{-\sqrt{b(0)}x}{\varepsilon}\right),$$
$$w_0(\tau) = [B - u_0(1)] \exp\left(\frac{-\sqrt{b(1)}(1-x)}{\varepsilon}\right).$$

The values of $\bar{A}$, $\bar{B}$ are given by

$$\bar{A} = \tilde{u}(k\varepsilon) \qquad \text{and} \qquad \bar{B} = \tilde{u}(1-k\varepsilon).$$

To solve the three BVPs (3.5),(3.6), (3.7),(3.8), and (3.9),(3.10), we use the difference scheme given in (2.19).

We repeat the procedure of solving the inner region problems (3.5),(3.6) and (3.7),(3.8), and the outer region problem (3.9),(3.10), by varying the value of k, until the solution satisfy the following relative error criteria,

$$\frac{|(u_i)^{n+1} - (u_i)^n|}{|(u_i)^n|} \le \delta, \tag{3.11}$$

where $(u_i)^n$ is the n^{th} iteration solution, and δ is the prescribed tolerance error bound.

REMARK 3.1. It is worthwhile to note that the domain decomposition is a nonoverlapping one, and the BVPs (3.5),(3.6), (3.7),(3.8), and (3.9),(3.10) have supplied with individual boundary conditions, and it opens the door for parallel processors to compute the numerical solution. By this way, one can reduce the computation time, almost one-third of the serial computer time, and obtain fourth-order accurate approximation.

4. STABILITY AND CONVERGENCE ANALYSIS

In this section, we derive results related to the stability of the continuous and discrete problems, and then the convergence of the method.

LEMMA 4.1. *Let v be a smooth function satisfying $v(0) \ge 0$, $v(1) \ge 0$, and $Lv(x) \le 0$, $\forall\, x \in D$. Then, $v(x) \ge 0$, $\forall\, x \in \bar{D}$. Further, we have the following uniform stability estimate,*

$$|v(x)| \le C\left[|v(0)| + |v(1)| + \max_{y\in\bar{D}} |f(y)|\right], \qquad \forall\, x \in \bar{D}.$$

PROOF. One can prove this result following the method given in [1]. ∎

THEOREM 4.2. *Let u be the solution of the BVP (2.1),(2.2), and u_i be the numerical solution obtained from the difference scheme (2.19). Then, we have*

$$|u(x_i) - u_i| \le Ch^4.$$

PROOF. Replacing the approximate solution $U = (u_1, \dots, u_{N-1})^t$ by the exact solution $\tilde{U} = (u(x_1), \dots, u(x_{N-1}))^t$ in (2.19), we obtain

$$\left(J - \frac{h^2}{60}BQ\right)\tilde{U} = C + \frac{h^2}{60}BR + \tilde{T}(h), \tag{4.1}$$

where $\tilde{T}(h) = (t_1(h), \dots, t_{N-1}(h))^t$ is the truncation error generated from this replacement.

The above system can be written in expanded form as

$$\begin{aligned} u(x_{j-2}) + 2u(x_{j-1}) - 6u(x_j) + 2u(x_{j+1}) + u(x_{j+2}) &= \frac{h^2}{20}\left[(q_{j-2}u(x_{j-2}) + r_{j-2})\right. \\ &+26(q_{j-1}u(x_{j-1}) + r_{j-1}) + 66(q_j u(x_j) + r_j) + 26(q_{j+1}u(x_{j+1}) \\ &\left.+r_{j+1}) + (q_{j+2}u(x_{j+2}) + r_{j+2})\right] + t_j(h), \qquad j = 2, \dots, N-2, \end{aligned} \tag{4.2}$$

$$\begin{aligned} 4u(x_0) - 7u(x_1) + 2u(x_2) + u(x_3) &= \frac{h^2}{12}\left[4(q_0 u(x_0) + r_0) + 41(q_1 u(x_1) + r_1)\right. \\ &\left.+14(q_2 u(x_2) + r_2) + q_3 u(x_3) + r_3\right] + t_1(h), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} u(x_{N-3}) + 2u(x_{N-2}) - 7u(x_{N-1}) + 4u(x_N) &= \frac{h^2}{12}\left[(q_{N-3}u(x_{N-3}) + r_{N-3})\right. \\ &+14(q_{N-2}u(x_{N-2}) + r_{N-2}) + 41(q_{N-1}u(x_{N-1}) + r_{N-1}) \\ &\left.+4(q_N u(x_N) + r_N)\right] + t_{N-1}(h). \end{aligned} \tag{4.4}$$

where the expression for truncation errors can be obtained by expanding each term in above equations about x_j using Taylor series expansion as

$$\begin{aligned} t_1(h) &= -\left(\frac{h^6}{48}\right)u^{(6)}(\xi_1), & x_0 < \xi_1 < x_3, \\ t_j(h) &= -\left(\frac{h^6}{120}\right)u^{(6)}(\xi_j), & x_{j-2} < \xi_j < x_{j+2}, \qquad j = 2, \dots, N-2, \\ t_{N-1}(h) &= -\left(\frac{h^6}{48}\right)u^{(6)}(\xi_{N-1}), & x_{N-3} < \xi_{N-1} < x_N. \end{aligned} \tag{4.5}$$

Subtracting (2.15)–(2.17) from (4.2)–(4.4), and denoting by $e_j = u(x_j) - u_j$, we get

$$\begin{aligned} e_{j-2} + 2e_{j-1} - 6e_j + 2e_{j+1} + e_{j+2} &= \frac{h^2}{20}\left[(q_{j-2}e_j) + 26(q_{j-1}e_{j-1}) + 66(q_j e_j) + 26(q_{j+1}e_{j+1})\right. \\ &\left.+ (q_j + 2e_{j+2})\right] + t_j(h), \qquad j = 2, \dots, N-2, \\ -7e_1 + 2e_2 + e_3 &= \frac{h^2}{12}\left[41q_1e_1 + 14q_2e_2 + q_3e_3\right] + t_1(h), \\ e_{N-3} + 2e_{N-2} - 7e_{N-1} &= \frac{h^2}{12}\left[q_{N-3}e_{N-3} + 14q_{N-2}e_{N-2} + 41q_{N-1}e_{N-1}\right] \\ &+ t_{N-1}(h). \end{aligned} \tag{4.6}$$

Letting $E = \tilde{U} - U$, we can express (4.6) in the following matrix form,

$$\left(J - \frac{h^2}{60}BQ\right)E = \tilde{T}(h) = O\left(h^6\right), \tag{4.7}$$

where $\tilde{T}(h) = (t_1(h), \dots, t_{N-1}(h))^t$.

Following the proof provided in Chawla and Subramanian [11], it can been shown that, for sufficiently small h,

$$\left\|\left(J-\frac{h^2}{60}BQ\right)^{-1}\right\| \le \|J^{-1}\| \le \left(\frac{1}{48h^2}+\frac{1}{12}\right), \tag{4.8}$$

where $\|\cdot\|$ is the matrix maximum norm, and from (4.7),

$$\|E\| \le \left\|\left(J-\frac{h^2}{60}BQ\right)^{-1}\right\| \left\|\tilde{T}(h)\right\|. \tag{4.9}$$

Thus, one can have

$$|u(x_i)-u_i| \le Ch^4, \tag{4.10}$$

where C is a constant independent of the mesh points x_i, and the step size h. It is worthwhile to note that the BVP (2.1),(2.2) does not have the singular perturbation parameter ε. ∎

PROPOSITION 4.3. *Let us consider the BVP,*

$$Lu(x) = f(x), \qquad x \in (c,d), \tag{4.11}$$

$$u(c) = \alpha, \qquad u(d) = \beta \tag{4.12}$$

and the same differential equation with a perturbation in the left- and right-hand side boundary conditions, that is to say, $u(c) = \alpha + O(\varepsilon)$, *and* $u(d) = \beta + O(\varepsilon)$. *We refer to the second problem as a perturbed BVP (PBVP). Let* u_1 *and* u_2 *be, respectively, the solutions of these problems. Then, we have the following estimate,*

$$|u_1(x) - u_2(x)| \le C\varepsilon, \qquad \forall x \in [c,d].$$

Hereafter, C *denotes a positive constant independent of the parameter* ε, *the mesh points* x_i, *and the step size* h.

PROOF. Let $w(x) = u_1(x) - u_2(x)$. Then, $w(x)$ satisfies the following BVP,

$$Lw(x) = 0, \qquad x \in (c,d),$$

$$w(c) = O(\varepsilon), \qquad w(d) = O(\varepsilon).$$

Applying Lemma 4.1 to the previous BVP, we get $|w(x)| \le C\varepsilon$. ∎

The following theorem is the main result of this article, which conveys the relation between the numerical solution using the transition boundary conditions and the exact solution.

THEOREM 4.4. *Let* u *be the solution of the BVP (4.11),(4.12) and* u_i *be the numerical solution of the respective PBVP by applying the difference scheme given in (2.19). Then,*

$$|u(x_i) - u_i| \le C\left(\varepsilon + h^4\right), \qquad \forall x \in [c,d].$$

PROOF. We have

$$|u(x_i) - u_i| \le |u(x_i) - u_2(x_i)| + |u_2(x_i) - u_i|,$$

where $u_2(x)$ is the solution of the perturbed BVP.

Applying Theorem 4.2 to the second part in the right-hand side of the above inequality, we get $|u_2(x_i) - u_i| \le Ch^4$. Combining this with the result of Proposition 4.3, we obtain the required estimate. ∎

By observing the facts that

$$\bar{A} = u(k\varepsilon) + O(\varepsilon)$$

and

$$\bar{B} = u(1-k\varepsilon) + O(\varepsilon),$$

where u is the solution of the BVP (1.1),(1.2), we obtain the following result.

THEOREM 4.5. *Let u be the solutions of the BVP (1.1),(1.2), and u_i be the numerical solution of one of the subdomain problems in the boundary layers or in the regular layer obtained by the difference scheme (2.19). Then,*

$$|u(x_i) - u_i| \le C\left(\varepsilon + h^4\right), \qquad \forall x_i \in [0,1].$$

5. NUMERICAL EXPERIMENTS

In this section, we provide some examples and the implementation of the numerical method. From the numerical tables, one can easily see the accuracy and performance of the method over other methods.

EXAMPLE 5.1. Consider the following singularly perturbed BVP,

$$-\varepsilon^2 u''(x) + u(x) = 0, \qquad x \in (0,1),$$
$$u(0) = 1, \qquad u(1) = 1.$$

The exact solution of this problem is

$$u(x) = \frac{(1-\exp(-1/\varepsilon))[\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon)]}{1-\exp(-2/\varepsilon)}.$$

Table 1. Errors correspond to the present scheme for Example 5.1.

$\varepsilon = 1.0e-02$		$\varepsilon = 1.0e-04$		$\varepsilon = 1.0e-06$	
Nodes	Error	Nodes	Error	Nodes	Error
0.0000e − 000	1.9543e − 015	0.0000e − 000	1.9543e − 015	0.0000e − 000	1.9543e − 015
2.5000e − 003	3.5467e − 007	2.5000e − 005	3.5467e − 007	2.5000e − 007	3.5467e − 007
1.5000e − 002	7.0957e − 007	1.5000e − 004	7.0957e − 007	1.5000e − 002	7.0957e − 007
2.7500e − 002	4.1704e − 007	2.7500e − 004	4.1704e − 007	2.7500e − 006	4.1704e − 007
4.0000e − 002	1.8063e − 007	4.0000e − 004	1.8063e − 007	4.0000e − 006	1.8063e − 007
5.2500e − 002	6.8873e − 008	5.2500e − 004	6.8873e − 008	5.2500e − 006	6.8873e − 008
6.5000e − 002	2.3244e − 008	6.5000e − 004	2.3244e − 008	6.5000e − 006	2.3244e − 008
7.7500e − 002	2.6634e − 009	7.7500e − 004	2.6634e − 009	7.7500e − 006	2.6634e − 009
1.0625e − 001	6.2744e − 006	3.2000e − 002	3.6370e − 005	1.2501e − 007	1.9452e − 006
2.3750e − 001	2.2626e − 008	1.8800e − 001	3.6035e − 007	2.500e − 007	6.6826e − 008
3.6875e − 001	1.0000e − 010	3.4400e − 001	5.3333e − 009	3.7500e − 007	2.2996e − 009
5.0000e − 001	8.8382e − 013	5.0000e − 001	1.5780e − 010	5.0000e − 007	1.5790e − 010
6.3125e − 001	1.0000e − 010	6.5600e − 001	5.3333e − 009	6.5625e − 007	5.3357e − 009
7.6250e − 001	2.2626e − 008	8.1200e − 001	3.6035e − 007	7.5000e − 007	6.6826e − 008
8.9375e − 001	6.2744e − 006	9.6800e − 001	3.6370e − 005	9.0625e − 007	4.5378e − 006
9.2250e − 001	2.6634e − 009	9.9923e − 001	2.6634e − 009	0.99999225	2.6634e − 009
9.3500e − 001	2.3244e − 008	9.9935e − 001	2.3244e − 008	0.99999350	2.3244e − 008
9.4750e − 001	6.8873e − 008	9.9948e − 001	6.8873e − 008	0.99999475	6.8873e − 008
9.6000e − 001	1.8063e − 007	9.9960e − 001	1.8063e − 007	0.99999600	1.8063e − 007
9.7250e − 001	4.1704e − 007	9.9973e − 001	4.1704e − 007	0.99999725	4.1704e − 007
9.8500e − 001	7.0957e − 007	9.9985e − 001	7.0957e − 007	0.99999850	7.0957e − 007
9.9750e − 001	3.5467e − 007	9.9998e − 001	3.5467e − 007	0.99999975	3.5467e − 007
1.0000e − 000	3.8743e − 013	1.0000e − 000	3.8743e − 013	1.0000e − 000	3.8743e − 013

Here, the reduced problem is a homogeneous algebraic equation, and it has the trivial solution, the transition boundary condition consists only the boundary layer correction terms, and it is given by

$$\tilde{u}(x) = \exp\left(\frac{-x}{\varepsilon}\right) + \exp\left(\frac{-(1-x)}{\varepsilon}\right).$$

We determine $\tilde{u}(x)$ at both the transition points $k\varepsilon$ and $1-k\varepsilon$.

Table 2. Maximum error of quintic spline scheme (2.19) for Example 5.1.

	Domain	$N = 64$	$N = 128$	$N = 256$	$N = 512$
	D_1	6.6240e − 06	1.3650e − 06	1.9410e − 07	2.1749e − 08
$\varepsilon = 2^{-10}$	D_2	2.6082e − 05	6.2291e − 06	1.2841e − 06	1.5926e − 07
	D_3	6.6240e − 06	1.3650e − 06	1.9410e − 07	2.1749e − 08
	D_1	6.6240e − 06	1.3650e − 06	1.9410e − 07	2.1749e − 08
$\varepsilon = 2^{-20}$	D_2	2.6472e − 05	6.6182e − 06	1.6545e − 06	4.1363e − 07
	D_3	6.6240e − 06	1.3650e − 06	1.9410e − 07	2.1749e − 08
	D_1	6.6240e − 06	1.3650e − 06	1.9410e − 07	2.1749e − 08
$\varepsilon = 2^{-30}$	D_2	2.6472e − 05	6.6182e − 06	1.6545e − 06	4.1363e − 07
	D_3	6.6240e − 06	1.3650e − 06	1.9410e − 07	2.1749e − 08

Table 3. Maximum error of the classical finite-difference scheme for Example 5.1.

	Domain	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\varepsilon = 2^{-10}$	$[0, 1]$	7.7521e − 02	1.8191e − 03	5.8032e − 04	1.7854e − 04
$\varepsilon = 2^{-20}$	$[0, 1]$	5.3580e − 03	1.8191e − 03	5.8032e − 04	1.7854e − 04
$\varepsilon = 2^{-30}$	$[0, 1]$	5.3580e − 03	1.8191e − 03	5.8032e − 04	1.7854e − 04

Table 4. Errors correspond to the present scheme for Example 5.2.

$\varepsilon = 1.0\text{e} - 02$		$\varepsilon = 1.0\text{e} - 04$		$\varepsilon = 1.0\text{e} - 06$	
Nodes	Error	Nodes	Error	Nodes	Error
0.0000e-000	4.6541e-017	0.0000e-000	4.6541e-017	0.0000e-000	4.6541e-017
2.5000e-003	3.5467e-007	2.5000e-005	3.5467e-007	2.5000e-007	3.5467e-007
1.5000e-002	7.0957e-007	1.5000e-004	7.0957e-007	1.5000e-006	7.0957e-007
2.7500e-002	4.1704e-007	2.7500e-004	4.1704e-007	2.7500e-006	4.1704e-007
4.0000e-002	1.8064e-007	4.0000e-004	1.8063e-007	4.0000e-006	1.8063e-007
5.2500e-002	6.8874e-008	5.2500e-004	6.8873e-008	5.2500e-006	6.8873e-008
6.5000e-002	2.3244e-008	6.5000e-004	2.3244e-008	6.5000e-006	2.3244e-008
7.7500e-002	2.6634e-009	7.7500e-004	2.6634e-009	7.7500e-006	2.6634e-009
1.8800e-001	7.7238e-006	1.8750e-001	3.6714e-007	1.8750e-001	3.6045e-007
3.4400e-001	4.4662e-008	3.4375e-001	5.4340e-009	3.4375e-001	5.3357e-009
5.0000e-001	3.6134e-009	5.0000e-001	1.5887e-010	5.0000e-001	1.5790e-010
6.5600e-001	4.4242e-008	6.5625e-001	5.3300e-009	6.5625e-001	5.3357e-009
8.1200e-001	7.6489e-006	8.1250e-001	3.6031e-007	8.1250e-001	3.6045e-007
9.2250e-001	2.3924e-009	9.9923e-001	2.3924e-009	0.99999225	2.3924e-009
9.3500e-001	2.0960e-008	9.9935e-001	2.0960e-008	0.99999350	2.0960e-008
9.4750e-001	6.0517e-008	9.9948e-001	6.0517e-008	0.99999475	6.0517e-008
9.6000e-001	1.5136e-007	9.9960e-001	1.5136e-007	0.99999600	1.5136e-007
9.7250e-001	3.1483e-007	9.9973e-001	3.1483e-007	0.99999725	3.1483e-007
9.8500e-001	3.5280e-007	9.9985e-001	3.5280e-007	0.99999850	3.5280e-007
9.9750e-001	1.5999e-006	9.9998e-001	1.5999e-006	0.99999975	1.5999e-006
1.0000e-000	3.1082e-015	1.0000e-000	3.1082e-015	1.0000e-000	3.1082e-015

The test problems are solved by using the step size $h = 1/96$ in the whole domain $D = (0, 1)$, more precisely, we took $h_1 = k\varepsilon/32$ in both the boundary layer regions D_1 and D_2, and $h_2 = (1 - 2k\varepsilon)/32$ in the regular region D_3. The numerical results are given in terms of errors in the tables for various values of ε and N. Table 1 shows the point-wise error for three values of ε.

Table 5. Maximum error of the numerical scheme given in [12] for Example 5.2.

	Domain	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
$\varepsilon=2^{-10}$	$[0,1]$	5.022e − 02	3.928e − 03	9.612e − 04	2.392e − 04	5.028e − 05	1.276e − 05
$\varepsilon=2^{-12}$	$[0,1]$	3.066e − 02	2.021e − 02	3.803e − 03	9.633e − 04	2.768e − 04	5.988e − 05
$\varepsilon=2^{-14}$	$[0,1]$	3.174e − 02	1.576e − 02	6.303e − 03	5.367e − 03	9.917e − 04	2.398e − 04
$\varepsilon=2^{-16}$	$[0,1]$	3.119e − 02	1.581e − 02	7.909e − 03	3.468e − 03	9.731e − 04	9.635e − 04
$\varepsilon=2^{-18}$	$[0,1]$	3.124e − 02	1.560e − 02	7.871e − 03	3.940e − 03	1.826e − 03	6.840e − 04
$\varepsilon=2^{-20}$	$[0,1]$	3.125e − 02	1.562e − 02	7.804e − 03	3.921e − 03	1.963e − 03	9.404e − 04
$\varepsilon=2^{-25}$	$[0,1]$	3.125e − 02	1.562e − 02	7.812e − 03	3.906e − 03	1.952e − 03	9.759e − 04
$\varepsilon=2^{-30}$	$[0,1]$	3.125e − 02	1.562e − 02	7.812e − 03	3.906e − 03	1.953e − 03	9.765e − 04

Table 6. Maximum error of quintic spline scheme (2.19) for Example 5.2.

	Domain	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-10}$	D_2	1.1085e − 06	4.3612e − 09	1.7166e − 10	7.6871e − 09	3.2306e − 09	1.3171e − 08
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-12}$	D_2	2.7369e − 05	7.9302e − 07	7.6023e − 09	5.4879e − 11	6.7166e − 10	2.2938e − 10
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-14}$	D_2	2.7938e − 05	8.3682e − 06	3.1935e − 07	4.2967e − 09	2.8891e − 09	2.7371e − 08
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-16}$	D_2	9.9238e − 05	2.0048e − 05	2.3285e − 05	1.0184e − 07	1.6093e − 09	1.0951e − 08
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-18}$	D_2	1.0430e − 05	2.4387e − 05	5.0912e − 06	6.7011e − 07	2.8982e − 08	2.1831e − 09
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-20}$	D_2	1.0550e − 04	2.6082e − 05	6.2291e − 06	1.2841e − 07	1.6197e − 07	7.7728e − 09
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-25}$	D_2	1.0588e − 04	2.6460e − 05	6.6062e − 06	1.6426e − 06	6.0279e − 07	9.1751e − 07
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
	D_1	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09
$\varepsilon^2=2^{-30}$	D_2	1.0588e − 04	2.6472e − 05	6.6178e − 06	1.6542e − 06	4.1323e − 07	1.1425e − 07
	D_3	3.4998e − 04	2.7289e − 05	1.3650e − 06	1.9410e − 07	2.1749e − 08	2.1553e − 09

In Table 2, the maximum errors in the subdomains D_1, D_2, and D_3 obtained from the quintic spline scheme given in (2.19) are presented. Table 3 shows the maximum pointwise error corresponds to the classical finite-difference scheme on piecewise uniform Shishkin meshes as given in Farrell *et al.* [1]. One can compare the results given in Tables 2 and 3, and easily observe that the present method produces accurate results.

EXAMPLE 5.2. Consider the nonhomogeneous self-adjoint SPP,

$$-\varepsilon^2 u''(x) + u(x) = -\cos^2(\pi x) - 2\varepsilon^2\pi^2\cos(2\pi x), \qquad x \in (0,1),$$
$$u(0) = 0, \qquad u(1) = 0.$$

The exact solution is given by

$$u(x) = \frac{[\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon)]}{1 - \exp(-1/\varepsilon)} - \cos^2(\pi x).$$

The transition boundary condition is given by

$$\tilde{u}(x) = -\cos^2(\pi x) + \exp\left(\frac{-x}{\varepsilon}\right) + \exp\left(\frac{-(1-x)}{\varepsilon}\right).$$

In Table 4, we have shown the maximum pointwise error for three different values of ε corresponds to Example 5.2. Here, for comparison we took the numerical results given in the recent paper [12]. The maximum pointwise error of [12] is presented in Table 5. For uniformity in the comparison, we determined the maximum pointwise error of the scheme (2.19) in each subdomain for the values of ε^2, and the results are given in Table 6. It is obvious that our method produces more accurate results than the method given in [12].

6. CONCLUSIONS

In this article, we proposed a numerical method for singularly perturbed reaction-diffusion problems, which combines the domain decomposition and the quintic spline difference scheme. First, we decompose the domain into three nonoverlapping subdomains (two boundary layer regions and one outer region), and making suitable BVPs in these subdomains, we apply the numerical scheme obtained from quintic spline. To determine the boundary conditions, we require the zero$^{\text{th}}$-order asymptotic approximate solution. We shall take the transition parameter as $k\varepsilon$, where $k = \ln(N)$, which is similar to the transition parameter used in the Shishkin-type meshes. By this suitable combination of the domain decomposition and the quintic spline difference schemes, our method enjoys the higher order convergence, provides special care for the boundary layer region problems, and open the door for parallel processors.

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